Why we like some investments more than others

Risk aversion is a psychological concept which describes how we are often willing to settle for a lower return in exchange for a greater certainty of outcomes or a decreased chance of losing money. Risk aversion is one of the key drivers when we are selecting investments. It also plays an important role in our decisions about leverage.

Defining Risk Aversion

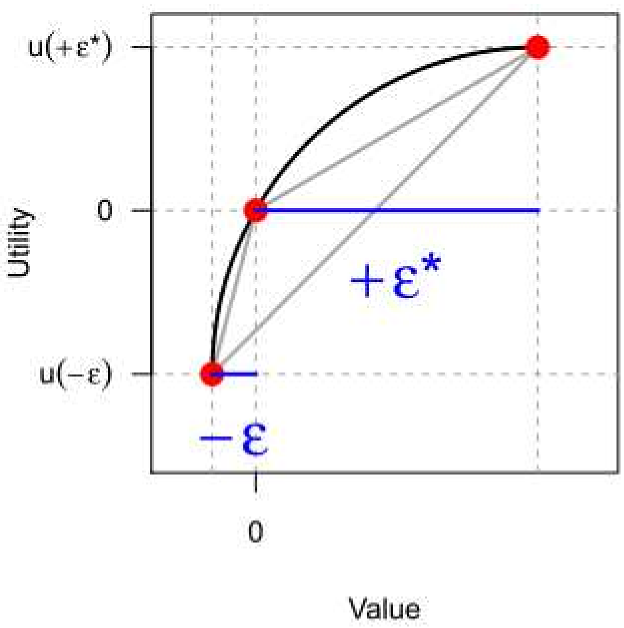
A systematic study of risk aversion usually starts with the assertion that we can summarize how individuals value money by acting as if they have a utility function mapping the value of money into the amount of pleasure they receive from the money.

For a utility function u(x) of money, Arrow and Pratt define two measures of risk aversion, the absolute and relative:

$$r(x) = -\frac{u''(x)}{u'(x)}$$

$$r^*(x) = -x\frac{u''(x)}{u'(x)}$$

These measures of risk aversion can be motivated as follows. An individual is asked if he or she is willing to accept a gamble with two equally-likely, randomly-chosen outcomes – ε and + ε*. Furthermore, if we fix – ε, we can adjust ε* until the individual is indifferent between accepting and rejecting the gamble. The more risk averse the individual is, the higher + ε* needs to be relative to – ε. We could also say that the more risk averse the individual, the flatter the slope of their utility curve is from 0 to + ε* relative to the slope from 0 to – ε. This, however, is the curvature of the utility function around zero. To compute the risk aversion, we also need to adjust for the size of – ε. To do this we standardize the curvature by the difference in utilities at + ε * and - ε. This is equivalent to the slope of the utility curve around zero. To ensure that a more risk averse individual has a higher measured risk aversion, we multiply this value by minus one. In summary, risk aversion is minus one times the curvature divided by the slope around zero.



Since we would like to measure an individual's risk aversion at all values of x, we consider only small changes in price around each x, and generalize the measure to arrive at the Arrow-Pratt measure of absolute risk aversion:

$$r(x) = -\frac{u''(x)}{u'(x)} \qquad \text{Eq. 1}$$

Here we use the second derivative of the utility function to measure curvature and the first derivative to standardize the measure.

We can also modify the argument above so risk aversion is measured relative the size of x to obtain their formula for relative risk aversion:

$$r*(x) = -x\frac{u''(x)}{u'(x)} \qquad \text{Eq. 2}$$

In this case, the curvature of the utility function decreases as value increases to maintain the same level of risk aversion. This models the case in which one is willing to take greater risks when the dollar values are greater.

Measuring one's risk aversion

Measuring one's risk aversion can be done several ways. I will concentrate on two methods. The first one assumes that the individual's absolute risk aversion is constant (known as the constant absolute risk aversion or CARA model). A variant of the first method could also be applied if we assumed the individual's relative risk aversion is constant (known as the constant relative risk aversion or CRRA model). The second method measures the individual's risk aversion at a single point on one's utility curve. The second method could be applied to multiple points on the utility curve to determine how risk aversion changes for different parts of the curve.

Both methods assume that data are collected by asking the individual when he or she is indifferent between a certain outcome x - z and a 50-50 gamble to receive either x + λ or x – λ. The certain outcome is adjusted by changing the value of z, so the individual is indifferent between the certain outcome and the gamble.

The first method assumes that r(x) on the right side of equation 1 is constant. Solving equation 1 for the CARA model, we know that u(x) must be of the general form

$$u(x) = 1 - e^{-\alpha x} \qquad \text{Eq. 3}$$

So we can solve for α in the equality

$$\tfrac{1}{2}e^{-\alpha(x-\lambda)} + \tfrac{1}{2}e^{-\alpha(x+\lambda)} = e^{-\alpha(x-z)} \qquad \text{Eq. 4}$$

Similarly, solving equation 2 for the CRRA model yields utility functions of the general form

$$u(x) = \frac{x^{1-\gamma}}{1-\gamma} \qquad \text{Eq. 5}$$

So we can solve for Υ in the equality

$$\tfrac{1}{2}\frac{(x-\lambda)^{1-\gamma}}{1-\gamma} + \tfrac{1}{2}\frac{(x+\lambda)^{1-\gamma}}{1-\gamma} = \frac{(x-z)^{1-\gamma}}{1-\gamma} \qquad \text{Eq. 6}$$

Solving for the parameters for the utility curves is easily done, but gives little insight into how one's preferences affect the calculated risk aversion measures. To see the relationship more clearly, we can use the second method for calculating one's risk aversion.

The second method calculates r(x) by assuming that the values of z and λ are small so the numerator and denominator of the absolute risk aversion formula can be approximated by their Taylor expansions.

From our calibration of z and λ we know that the following utilities are equal assuming the individual interprets the probabilities objectively:

$$\tfrac{1}{2}u(x-\lambda)+\tfrac{1}{2}u(x+\lambda)=u(x-z) \qquad \text{Eq. 7}$$

Expanding both sides using a Taylor series approximation around x yields

$$u(x)-\tfrac{1}{2}\lambda u'(x)+\tfrac{1}{2}\lambda u'(x)+\tfrac{1}{2}\lambda^2 u''(x)\approx u(x)-zu'(x)+\tfrac{1}{2}z^2u''(x)$$

This simplifies to

$$\tfrac{1}{2}\lambda^2 u''(x)-\tfrac{1}{2}z^2u''(x)\approx -zu'(x)$$

Resulting in a calculated absolute risk aversion at x of

$$r(x)=-\frac{u''(x)}{u'(x)}\approx\frac{2z}{\left(\lambda^2-z^2\right)} \qquad \text{Eq. 8}$$

The relative risk aversion is

$$r(x)=-x\frac{u''(x)}{u'(x)}\approx\frac{2zx}{\left(\lambda^2-z^2\right)} \qquad \text{Eq. 9}$$

Computing the certainty-equivalent return from a return distribution

This argument can be expanded beyond just two outcomes for the gamble. If for instance, we obtain the certainty equivalent for a distribution of outcomes, where ε has mean zero and variance σ^2, we have the equation

$$E[u(x+\varepsilon)]=u(x-z)$$

By expanding both sides using a Taylor series approximation,

$$E\left[u(x)+\varepsilon u'(x)+\tfrac{1}{2}\varepsilon^2u''(x)\right]\approx u(x)-zu'(x)+\tfrac{1}{2}z^2u''(x)$$

Which becomes

$$u(x)+E[\varepsilon]u'(x)+\tfrac{1}{2}E\left[\varepsilon^2\right]u''(x)\approx u(x)-zu'(x)+\tfrac{1}{2}z^2u''(x)$$

$$\tfrac{1}{2}\sigma^2u''(x)-\tfrac{1}{2}z^2u''(x)\approx -zu'(x)$$

$$r(x) = -\frac{u''(x)}{u'(x)} \approx \frac{2z}{(\sigma^2 - z^2)} \qquad \text{Eq. 10}$$

If we only use the first derivative on the right hand side to derive equation 10 by assuming that z is small relative to σ, then the z does not appear in the denominator and the equation can be rewritten as

$$x - z \approx x - \tfrac{1}{2} r(x) \sigma^2 \qquad \text{Eq. 11}$$

This allows us to calculate the certainty-equivalent return, x – z, for a return stream with mean x and variance σ^2 along with the absolute risk aversion. Equation 11 is also often called the CFA utility formula.

Using a similar derivation assuming a constant relative risk aversion yields

$$r^*(x) = -x\frac{u''(x)}{u'(x)} \approx \frac{2zx}{(\sigma^2 - z^2)} \qquad \text{Eq. 13}$$

When z is removed from the denominator, this becomes

$$x - z \approx x - \tfrac{1}{2} r^*(x) \frac{\sigma^2}{x} \qquad \text{Eq. 14}$$

This allows us to calculate the certainty-equivalent return, x - z, from the mean and variance of the return stream assuming constant relative risk aversion.